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EDGE EFFECTS IN THE STRESS STATE OF A THIN ELASTIC INTERLAYER

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The edge effects in the stress state of an interlayer in stretching and shearing by rigid slabs are studied. On the basis of the equations of momentless and moment elastic layers, we solve problems modeling qualitatively the stress-strain state in the "soft" layer between two "rigid" layers.

High stress gradients (edge or interlayer effects) occurring at interlayer surfaces in the neighborhood of the free boundaries of laminated structures can cause delamination and failures at the boundaries of the layers whose mechanical properties differ greatly (at the boundaries of rigid and soft layers). The interlayer effects cannot be described by the classical equations of the plate and shell theory or by the models of laminated plates in which the longitudinal stresses in soft layers are ignored. Blumberg and Tamuzh [1] investigated the edge effects in soft and rigid layers of a laminated plate on the basis of the equations of the Ambartsumyan theory of anisotropic plates and the Kirchhoff–Love equations, respectively. The edge effect is studied by the method of boundary functions. Most numerical algorithms to determine the interlayer stresses are based on the finite-element method.

In this paper, we study the stress state of a thin elastic interlayer between two nondeformable slabs in tension and shear under conditions of plane stress and strain.

Ivanov [2] developed a technique for constructing the equations of an elastic layer, which is based on the expansion of displacements and stresses in the planar problem of the theory of elasticity in terms of the Legendre polynomials. For each of the desired functions (the displacements and stresses), several approximations are used. The first-approximation equations of the elastic layer (the equations of the moment layer) and their general solutions were derived by the authors in [3]. The first-approximation equations are formulated in terms of the quantities usual for the theory of plates: the displacements averaged over the thickness, the angles of rotation, forces, and moments. Similarly, one can construct the equations of a momentless layer in terms of the displacements and forces averaged over the thickness. Omitting the derivation of the equations used in our further consideration, we discuss briefly their principal properties.

If the thickness of the layer h is small, by virtue of the Saint-Venant principle, the boundary conditions at its ends can be divided into two groups: the conditions that affect the solution for all $|x_1| \leqslant l$, which we call the basic boundary conditions, and the conditions that affect the solution only in the neighborhood of the cross sections $x_1 = \pm l$ (Fig. 1). In constructing the one-dimensional layer equations, we require that the boundary-value problem be solvable for any basic boundary conditions and the order of the differential equations be independent of the type of boundary condition specified at its surfaces (this allows us to specify correctly the conjugation conditions at the interlayer surfaces).

1. Equations of the Momentless Elastic Layer. We write the system of momentless-layer equations in the form

$$\frac{d\sigma_{11}^0}{dx_1} + \frac{6\mu}{h^2}(u_1^+ - u_1^-) - \frac{12\mu}{h^2}u_1^0 = 0, \quad \frac{d\sigma_{21}^0}{dx_1} + \frac{6a}{h^2}(u_2^+ + u_2^-) - \frac{12a}{h^2}u_2^0 = 0,$$

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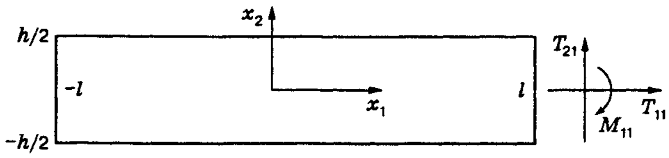

Fig. 1

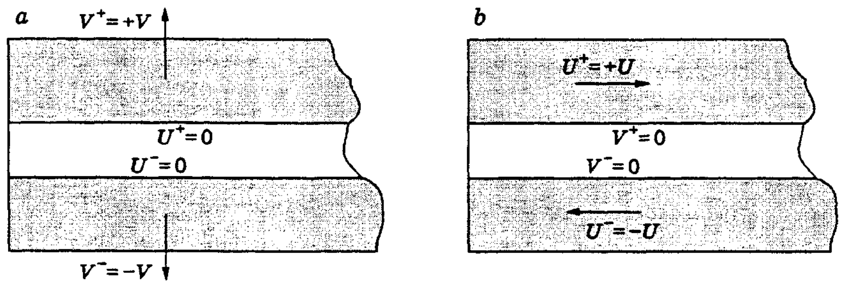

Fig. 2

$$
\begin{gathered}
\sigma_{11}^0 - a\,\frac{du_1^0}{dx_1} - \frac{b}{h}\,(u_2^+ - u_2^-) = 0, \quad \sigma_{21}^0 - \mu\,\frac{du_2^0}{dx_1} - \frac{b}{\mu}\,(u_1^+ - u_1^-) = 0, \\
\sigma_{12}^+ - \sigma_{12}^- = \frac{12\mu}{h}\Big[\frac{1}{2}(u_1^+ + u_1^-) - u_1^0\Big], \quad \sigma_{12}^+ + \sigma_{12}^- = 2\sigma_{12}^0, \\
\sigma_{22}^+ - \sigma_{22}^- = \frac{12a}{h}\Big[\frac{1}{2}(u_2^+ + u_2^-) - u_2^0\Big], \quad \sigma_{22}^+ + \sigma_{22}^- = 2\Big[\frac{a}{h}\,(u_2^+ - u_2^-) + b\,\frac{du_1^0}{dx_1}\Big], \\
a = \frac{E(1-\nu)}{(1-\nu)(1-2\nu)}, \qquad b = \frac{E\nu}{(1-\nu)(1-2\nu)},
\end{gathered}
\tag{1.1}
$$

where u_i^0 and σ_{ij}^0 are the displacements and stresses averaged over the thickness, μ is the shear modulus, $u_i^{\pm}$ and $\sigma_{ij}^{\pm}$ are the displacements and the stresses for $x_2 = \pm h/2$, E is the Young's modulus, and ν is the Poisson ratio.

Tension of the Layer. For tension of the interlayer (the light part in Fig. 2a), the conditions at the layer surfaces and the boundary conditions have the form

$$
\begin{gathered}
u_1^+ = u_1^- = 0, \qquad u_2^{\pm} = \pm V, \\
\sigma_{11}^0 = \sigma_{21}^0 = 0 \quad \text{for} \quad x_1 = 0, \qquad u_1^0,\ u_2^0 \to 0 \quad \text{for} \quad x_1 \to \infty.
\end{gathered}
\tag{1.2}
$$

From (1.1) and (1.2), we obtain

$$
\sigma_{21}^0 = u_2^0 = 0, \quad \sigma_{22}^+ = \sigma_{22}^- = \frac{2a}{h}\,V + b\,\frac{du_1^0}{dx_1}, \quad \sigma_{12}^- = -\sigma_{12}^+ = 6\mu\,\frac{1}{h}\,u_1^0,
$$

and the determination of the displacements and stresses in the interlayer reduces to the problem

$$
\begin{gathered}
\frac{d\sigma_{11}^0}{dx_1} - \frac{12\mu}{h^2}\,u_1^0 = 0, \qquad \sigma_{11}^0 - a\,\frac{du_1^0}{dx_1} - 2b\,\frac{V}{h} = 0, \\
\sigma_{11}^0 = 0 \quad \text{for} \quad x_1 = 0, \qquad u_1^0 \to 0 \quad \text{for} \quad x_1 \to \infty.
\end{gathered}
\tag{1.3}
$$

From (1.3) we obtain

$$
u_1^0 = \frac{2bV}{a\omega_1}\exp\Big(-\frac{\omega_1 x_1}{h}\Big), \qquad \omega_1 = 2\sqrt{\frac{3\mu}{a}},
$$

and, consequently,

$$\sigma_{22}^{+} = \sigma_{22}^{-} = (2a/h)V[1 - (b/a)^2 \exp(-\omega_1 x_1/h)],$$
$$\sigma_{12}^{+} = -\sigma_{12}^{-} = 2\sqrt{3\mu a}\, bV \exp(-\omega_1 x_1/h)/(ha).$$

Far from the edge $x_1 = 0$, the uniform stress state, which corresponds to the uniaxial deformation, occurs in the interlayer. For this state, we have

$$\sigma_{11}^{0} = 2bV/h, \quad \sigma_{22}^{0} = 2aV/h, \quad \sigma_{21}^{0} = 0, \quad \tau_{\max} = (\sigma_{22}^{0} - \sigma_{11}^{0})/2 = 2\mu V/h.$$

In the vicinity of the edge $x_1 = 0$, the stress state is nonuniform. Significant shear stresses occur at the surfaces $x_2 = \pm h/2$. Their maximum value is

$$2\sqrt{3\mu a}\, bV/(ha) = 2\sqrt{6}\,\mu\nu V/(\sqrt{(1-\nu)(1-2\nu)}\, h)$$

and, for $\nu = 1/3$, it exceeds the maximum shear stress $\tau_{\max}$ occurring in the zone of uniform deformation by a factor of 3.5.

Shear of the Interlayer. In the case of shearing the interlayer (Fig. 2b), we have

$$u_1^{\pm} = \pm W, \qquad u_2^{+} = u_2^{-} = 0,$$
$$\sigma_{11}^{0} = \sigma_{21}^{0} = 0 \quad \text{for} \quad x_1 = 0, \qquad \sigma_{11}^{0},\, u_2^{0} \to 0 \quad \text{for} \quad x_1 \to \infty. \tag{1.4}$$

From (1.1) and (1.4), we obtain

$$\sigma_{12}^{+} - \sigma_{12}^{-} = -12\mu u_1^{0}/h, \qquad \sigma_{12}^{+} + \sigma_{12}^{-} = 2\sigma_{21}^{0},$$
$$\sigma_{22}^{+} - \sigma_{22}^{-} = -12 a u_2^{0}/h, \qquad \sigma_{22}^{+} + \sigma_{22}^{-} = 2b\,\frac{du_1^{0}}{dx_1},$$

and determination of the displacements and stresses in the interlayer reduces to solving the problem

$$\frac{d\sigma_{11}^{0}}{dx_1} - \frac{12\mu}{h^2}u_1^{0} = 0, \qquad \sigma_{11}^{0} - a\,\frac{du_1^{0}}{dx_1} = 0,$$
$$\sigma_{11}^{0} = 0 \quad \text{for} \quad x_1 = 0, \qquad \sigma_{11}^{0} \to 0 \quad \text{as} \quad x_1 \to \infty; \tag{1.5}$$

$$\frac{d\sigma_{21}^{0}}{dx_1} - \frac{12a}{h^2}u_2^{0} = 0, \qquad \sigma_{21}^{0} - \mu\,\frac{du_2^{0}}{dx_1} - 2\mu\,\frac{W}{h} = 0,$$
$$\sigma_{21}^{0} = 0 \quad \text{for} \quad x_1 = 0, \qquad u_2^{0} \to 0 \quad \text{as} \quad x_1 \to \infty. \tag{1.6}$$

From (1.5) and (1.6), with allowance for (1.4) we obtain

$$\sigma_{11}^{0} = 0, \qquad u_1^{0} = 0, \qquad \sigma_{21}^{0} = 2\mu W(1 - \exp(-\omega_2 x_1/h))/h,$$
$$u_2^{0} = 2W \exp(-\omega_2 x_1/h)/\omega_2, \qquad \omega_2 = 2\sqrt{3a/\mu} = 2\sqrt{6(1-\nu)/(1-2\nu)}$$

and, consequently, $\sigma_{22}^{+} = -\sigma_{22}^{-} = 12aW\exp(-\omega_2 x_1/h)/(h\omega_2)$ and $\sigma_{12}^{+} = \sigma_{12}^{-} = \sigma_{21}^{0}$. Thus, under the conditions (1.4), the uniform state of the pure shear stress is realized in the interlayer at a large distance from the edge $x_1 = 0$. In the vicinity of the edge $x_1 = 0$, the stress state, which is characterized by significant tensile stresses occurring at the surface $x_2 = -h/2$, is nonuniform. We note that

$$\omega_2 = \frac{2(1-\nu)}{1-2\nu}\,\omega_1,$$

and, consequently, the edge-effect zone when the interlayer is shifted is much smaller than that when the interlayer is in tension.

2. Equations of a Moment Elastic Layer. If the displacements are prescribed at the layer surfaces, the equations of a moment elastic layer can be reduced to the following system of differential equations [3]:

$$\eta^2 u_0'' - \frac{3(1-\gamma)}{2} u_0 = -\frac{3(1-\gamma)}{4}(u^+ + u^-) - \frac{\gamma\eta}{2}(v^+ - v^-)',$$
$$2\eta^2 u_1'' - 15(1-\gamma)u_1 - 6\gamma\eta v_0' = -\frac{15(1-\gamma)}{2}(u^+ - u^-) - 3\gamma\eta(v^+ + v^-)', \tag{2.1}$$
$$(1-\gamma)\eta^2 v_0'' + 2\gamma\eta u_1' - 6v_0 = -\frac{\eta(1-\gamma)}{2}(u^+ + u^-) - 3(v^+ + v^-) + (1-\gamma)f.$$

The forces and the moments are expressed through u_0, u_1, and v_0 by the formulas

$$t_{11} = \alpha(\eta u_0' + \gamma(v^+ - v^-)/2), \quad m_{11} = \alpha(\eta u_1' - 3\gamma v_0 + 3\gamma(v^+ + v^-)/2),$$
$$t_{22} = \alpha(\gamma\eta u_0' + (v^+ - v^-)/2), \quad m_{22} = \alpha(\gamma\eta u_1' - 3v_0 + 3(v^+ + v^-)/2),$$
$$t_{12} = \eta v_0' + (u^+ - u^-)/2, \qquad m_{12} = -3u_0 + 3(u^+ + u^-)/2, \tag{2.2}$$
$$\sigma_{12}^{\pm} = t_{12} \pm m_{12} + r_{12}, \quad \sigma_{22}^{\pm} = t_{22} \pm m_{22}, \quad r_{12} = -5u_1 + 5(u^+ - u^-)/2.$$

For the plane strain and stress, we have $\gamma = \nu/(1-\nu)$ and $\gamma = \nu$, respectively.

Equations (2.1) and (2.2) can be written in dimensionless variables

$$\bar{\sigma}_{ij} = \frac{\sigma_{ij}}{\sigma_0}, \quad \bar{\varepsilon}_{ij} = \frac{\varepsilon_{ij}}{\varepsilon_0}, \quad \varepsilon_0 = \frac{\sigma_0}{\mu}, \quad \bar{u}_i = \frac{2u_i}{h\varepsilon_0}, \quad \xi = \frac{x_1}{L_0}, \quad \zeta = \frac{2x_2}{h}, \quad \eta = \frac{h}{2L_0},$$
$$f_i^0 = \frac{f_i^0 h}{2\sigma_0}, \quad \sigma_{11}^0 = \frac{1}{h}T_{11} = \frac{T_{11}}{h\sigma_0}, \quad \sigma_{21}^0 = \frac{1}{h}T_{21} = t_{21} = t_{12} = \frac{T_{21}}{h\sigma_0}, \quad m_{11} = \frac{6M_{11}}{h^2\sigma_0},$$
$$u_0 = \frac{1}{h\varepsilon_0}\int_{-h/2}^{h/2} \frac{u_1}{h}\,dx_2, \qquad u_1 = \frac{6}{h^2\varepsilon_0}\int_{-h/2}^{h/2} \frac{u_1}{h} x_2\,dx_2, \qquad v_0 = \frac{1}{h\varepsilon_0}\int_{-h/2}^{h/2} \frac{u_2}{h}\,dx_2,$$

where σ_0 is the characteristic stress. The dimensional quantities are given in Fig. 1.

Tension of the Interlayer. The following conditions are specified at the layer surfaces:

$$u_1^+ = u_1^- = 0, \qquad u_2^{\pm} = \pm V. \tag{2.3}$$

The boundary conditions have the form

$$t_{11}(0) = t_{21}(0) = m_{11}(0) = 0. \tag{2.4}$$

We assume that the uniform stress state is realized at the distance from the edge $\xi = 0$.

From (2.1) and (2.3), we obtain

$$u_0 \to 0, \quad u_1 \to 0, \quad v_0 \to 0, \quad t_{11} \to \alpha\gamma V, \quad t_{22} \to \alpha V \quad \text{for} \quad \xi \to \infty. \tag{2.5}$$

In the neighborhood of the edge $\xi = 0$, the stress state is nonuniform. It follows from (2.1)–(2.5) that

$$u_1 \equiv 0, \qquad v_1 \equiv 0, \qquad t_{12} \equiv 0, \qquad r_{12} \equiv 0,$$
$$u_0 = \frac{\gamma V}{\eta\omega_1}\exp\Big(-\frac{\omega_1\xi}{\eta}\Big), \qquad t_{11} = \frac{2\gamma V}{1-\gamma}\Big[1 - \exp\Big(-\frac{\omega_1\xi}{\eta}\Big)\Big], \qquad \omega_1 = \sqrt{\frac{3(1-\gamma)}{2}}.$$

At the layer surfaces $\zeta = \pm 1$, the following normal and shear stresses occur:

$$\sigma_{22}^{\pm} = \frac{2V}{1-\gamma}\Big[1 - \gamma^2\exp\Big(-\frac{\omega_1\xi}{\eta}\Big)\Big], \qquad \sigma_{12}^{\pm} = \mp\frac{3\gamma V}{\omega_1}\exp\Big(-\frac{\omega_1\xi}{\eta}\Big).$$

At the distance from the edge $\xi = 0$, the uniform stress state

$$t_{11} = \frac{2\gamma V}{1-\gamma}, \qquad t_{22} = \frac{2V}{1-\gamma}$$

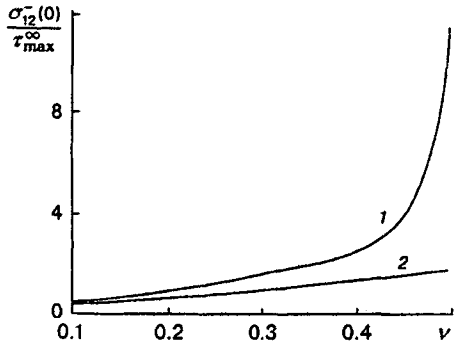

Fig. 3

is realized. In this case, the maximum shear stress at infinity is

$$\tau^\infty_{\max} = \frac{1}{2}(t_{22} - t_{11}) = V$$

and acts on the site which makes an angle of 45° with the x axis.

Thus, we have $\sigma^\pm_{12}(0)/\tau_{\max} = \mp 3\gamma/\omega_1$. The parameter ω_1 characterizes the extent of the edge effect.

The curve of $\sigma^-_{12}(0)/\tau^\infty_{\max}$ versus ν is shown in Fig. 3. The strong dependence of the shear stress which occurs in the neighborhood of the free edge upon tension of the interlayer on the Poisson ratio for plane strains (curve 1) is explained by the fact that the strain in the direction perpendicular to the cross section of the layer vanishes ($\varepsilon_3 = 0$). For plane strain, the shear stresses in the proximity of the free surface can significantly exceed the shear stresses which correspond to the uniform stress state (curve 2).

Shear of the Interlayer. The following conditions are specified at the layer surface:

$$u^+ = U, \qquad u^- = U, \qquad v^+ = 0, \qquad v^- = 0. \tag{2.6}$$

The boundary conditions for $\xi = 0$ have the form

$$t_{11}(0) = t_{21}(0) = m_{11}(0) = 0. \tag{2.7}$$

The uniaxial stress state which corresponds to the pure shear is realized at infinity:

$$u_0(\infty) = v_0(\infty) = t_{11}(\infty) = 0, \qquad u_1(\infty) = U. \tag{2.8}$$

From (2.1) and (2.6)–(2.8), we obtain

$$u_0 \equiv 0, \qquad u_1 = \exp\Big(-\frac{\omega_2\xi}{\eta}\Big)\Big(C_1\cos\frac{\beta}{\eta}\xi + C_2\sin\frac{\beta}{\eta}\xi\Big) + U,$$

$$v_0 = -\exp(-\omega_2\xi/\eta)[C_1(b_1\cos\beta\xi - b_2\sin\beta\xi) + C_2(b_2\cos\beta\xi + b_1\sin\beta\xi)],$$

where

$$b_1 = \frac{A(15\gamma + 2(A^2+B^2) - 15)}{6\gamma(A^2+B^2)}, \qquad b_2 = \frac{B(15\gamma - 2(A^2+B^2) - 15)}{6\gamma(A^2+B^2)},$$

$$\omega_2 = \frac{\sqrt{6}}{4}\sqrt{-\gamma + 4\sqrt{5} + 9}, \qquad \beta = \frac{\sqrt{6}}{4}\sqrt{\gamma + 4\sqrt{5} - 9}, \qquad C_1 = \frac{U(\eta\beta + 3b_2)}{\Delta},$$

$$C_2 = \frac{U(3b_1\gamma - \eta\omega_2)}{\Delta}, \qquad \Delta = -\eta[\eta b_2(\omega_2^2 + \beta^2) + 3\gamma\beta(b_1^2 + b_2^2)].$$

Using formulas (2.2), one can calculate the stresses at the layer surface: $\sigma^\pm_{12} = \eta v_0' - 5u_1 + 6U$ and $\sigma^+_{22} = -\sigma^-_{22} = \alpha(\gamma\eta u_1' - 3v_0)$. In the neighborhood of the edge $\xi = 0$, the stress state is nonuniform, the

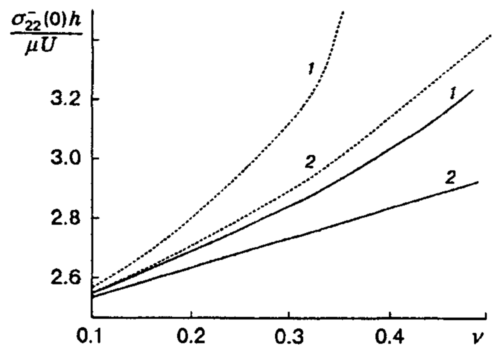

Fig. 4

normal stresses at the layer surfaces having the form

$$\sigma_{22}^{\pm}(0) = \pm\frac{6(\gamma+1)U}{\Delta}[\eta(\beta b_1 - \omega_2 b_2) + 6\gamma b_1 b_2].$$

At the surface $\zeta = -1$, these are the tensile stresses which can lead to separation of the layer.

Figure 4 shows the normal stress $\sigma_{22}^{-}(0)h/(\mu U)$ versus the Poisson ratio ν (curve 1 refers to the plane strains, and curve 2 to the plane stresses; dashed curves refer to the equations of a momentless layer, and solid curves to the equations of a moment layer).

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